A Unified Treatment of Quasi-Exactly Solvable Potentials I

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A set of quasi-exactly solvable quantum mechanical potentials associated with the Pöschl-Teller potential, the generalized Pöschl-Teller potential, the Scarf potential, and the harmonic oscillator potential have been studied. Solutions of the Schrödinger equation for each potential have been determined and the eigenstates are expressed in terms of the orthogonal polynomials. The potentials are related to each other by suitable change of variables.

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INTRODUCTION

The discovery of new class of physically significant spectral problems, called quasi-exactly solvable(QES) models[1, 2], has attracted much attention[3, 4]. Several methods[5, 6] for the generation of the QES potentials have been worked out and consequently a number of QES potentials have been suggested. One of the methods for the calculation of eigenstates and eigenvalues of the QES potentials is the use of orthogonal polynomials. Bender and Dunne have showed[7] that there is a correspondence between the QES models in quantum mechanics and the set of the orthogonal polynomials $P_m(E)$, which are polynomials in energy E. In particular they have demonstrated the properties of the polynomials for the QES sextic oscillator potential[8]. Other aspects of the $P_m(E)$ have been discussed in various papers[9, 10]. In this paper we show that the polynomial $P_m(E)$ normally appears as a factor of the coefficient in the expansion of the eigenstates of the Schrödinger equation in the presence of the QES Pöschl-Teller potential, the Generalized QES Pöschl-Teller potential and the PT symmetric Scarf potential. We also show that the $P_m(E)$ satisfies the same three-term recurrence relations for all three potentials.

An algorithm generating the analytical eigenfunctions as well as the eigenvalues of the Schrödinger equation for various QES potentials is developed. The procedure presented here reproduces the results of the exactly solvable Schrödinger equations in proper limits. Our formalism signals that, there is a closer similarity between QES Pöschl-Teller potential and the perturbed Pöschl-Teller potential[11, 12]. It is well known that the harmonic oscillator and the Pöschl-Teller potential possess the similar theoretical behavior in a certain range of variable. In contrast to enormous interest in the perturbed harmonic oscillator, there have been only a number of studies on the perturbed Pöschl-Teller potential[11]

We show that the QES Pöschl-Teller potential can be transformed on to the generalized QES Pöschl-Teller potential and the QES PT symmetric Scarf potential by replacing coordinate $x \to x/2$ and $x \to x/2 + i\pi/4$, respectively. These potentials are also related to the QES harmonic oscillator potential by redefining the parameters and applying an appropriate limiting procedure.

The paper is organized as follows. In section 2 we discuss the generation and solution of the QES Pöschl-Teller potential. The QES generalized Pöschl-Teller potential and the QES PT symmetric Scarf potentials are obtained from the QES Pöschl-Teller potential in sections 3 and 4, respectively. Transformations of QES Pöschl-Teller potential to the sextic oscillator potential and the QES PT symmetric Scarf potential to the PT symmetric anharmonic oscillator potential are discussed in sections 5 and 6 respectively. In conclusive remarks we discuss the implementation of our method on the other QES potentials which will be the topic of another publication.

QES PÖSCHL-TELLER POTENTIAL

The QES Pöschl-Teller potential can be generated by several methods[3, 11, 13]. One method is to use the Lie algebraic technique. The linear and bilinear combinations of the operators of the sl(2,R)[18] Lie algebra with the

standard realization[3, 4, 13], leads to the following differential equation,

$$z(1-z)\frac{d^2\Re_j(z)}{dz^2}+(L+\frac{3}{2}+z(B+4j-qA^2z))\frac{d\Re_j(z)}{dz}-(\lambda-2jqA^2z)\Re_j(z)=0 \tag{1}$$

where L, q, A and λ are constants and $j=0,1/2,1,\cdots$. The differential equation(1) becomes QES, provided B is taken as:

$$B=-\frac{1}{2}\left(2L+8j+5-\sqrt{1+4A(A+1+(2L+8j+5)qA)}\right). \tag{2}$$

Moreover, it becomes exactly solvable with the condition $q=0$. The function $\Re_j(z)$ is a polynomial of degree $2j$. In order to obtain the QES quantum mechanical potentials we can transform (1) in the form of Schrödinger equation by introducing the variable

$$z=-\sinh^2\alpha x. \tag{3}$$

Now we define the wave function as follows:

$$\psi(x)=(\cosh\alpha x)^{qA^2-B-L-4j-2}(\sinh\alpha x)^{1+L}e^{-\frac{1}{4}qA^2\cosh 2\alpha x}\Re_j(-\sinh^2\alpha x). \tag{4}$$

Substituting (3) and (4) in (1) we obtain the Schrödinger equation with ($\hbar=2m=1$):

$$-\frac{d^2\psi(x)}{dx^2}+(V(x)-E)\psi(x)=0 \tag{5}$$

where the potential $V(x)$ is given by

$$\begin{aligned} V(x) &= L(L+1)\alpha^2\csc h^2\alpha x-A(A+1)\alpha^2\sec h^2\alpha x+\\ &\quad q(2BA^2\sinh^2\alpha x+qA^4\sinh^4\alpha x)\alpha^2\tanh^2\alpha x \end{aligned} \tag{6}$$

The eigenvalues of energy are given by the expression

$$E=\left[-(L-A+2m+1)^2+(L-A+B+4j+2)(2L+4m+3)+4\lambda\right]\alpha^2. \tag{7}$$

One can check that for $q=0$ the potential given in (6) is exactly solvable Pöschl-Teller potential and the eigenstates of the Schrödinger equation can be expressed in terms of the Jacobi polynomials. It is easy to see that the equalities $B=A-L-4j-2$ and $\lambda=4m(L-M+m+1)$ hold when $q=0$. Then the eigenvalues of the Schrödinger equation take the form

$$E=-\alpha^2\left[(L-A+2m+1)^2+4m(L-M+m+1)\right]. \tag{8}$$

Next task is now to determine the eigenfunction of the QES Schrödinger equation(5). Therefore, we search for a solution of (1) by substituting the polynomial

$$\Re_j(z)=\sum_{m=0}^{2j}a_m z^m \tag{9}$$

which leads to an expression for the coefficients a_m

$$a_m=\frac{(4qA^2)^m(2j)!(2L+1)!(L+m)!}{2m!(2j-m)!(2L+1+2m)!}P_m(\lambda). \tag{10}$$

The polynomial $P_m(\lambda)$ satisfies the following three-term recurrence relation

$$2(2j-m)qA^2P_{m+1}(\lambda)+m(2L+2m+1)P_{m-1}(\lambda)-2(\lambda+m(B+4j-m+1))P_m(\lambda)=0 \tag{11}$$

with the initial condition $P_0(\lambda)=1$. The polynomial $P_m(\lambda)$ vanishes for $m\geqslant 2j+1$ and the roots of $P_{2j+1}(\lambda)=0$ corresponds to the $\lambda-$eigenvalues of the Schrödinger equation(5). The first three of these polynomials are given by

$$\begin{aligned} P_1 &= \lambda\\ P_2 &= \lambda^2-(B+4j)\lambda-j(2L+3)qA^2\\ P_3 &= \lambda^3-(3B12j-4)\lambda^2+\\ &\quad \left[2B(B-2)+16j(B+2j-1)+(2L-j(6L+13)+5)qA^2\right]\lambda+\\ &\quad 2j(2L+3)(B+4j-2)qA^2 \end{aligned} \tag{12}$$

The recurrence relation (11)can also be put in the matrix form. The tridiagonal matrix characterizes the system,

$$\begin{pmatrix} \beta_0-4\lambda & \mu_{2j} & & & \\ \gamma_1 & \beta_1-4\lambda & \mu_{2j-1} & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{2j-1} & \beta_{2j-1}-4\lambda & \mu_1 \\ & & & \gamma_{2j} & \beta_{2j}-4\lambda \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_{2j-1} \\ P_{2j} \end{pmatrix} = 0 \tag{13}$$

where the parameters in matrix elements are given by

$$\gamma_m = 2m(2L+2m+1), \quad \mu_m = 4mqA^2, \quad and \quad \beta_m = 4m(B+4j-m+1). \tag{14}$$

Analytical solutions of the recurrence relation (11) and the determinant of (13) for λ are available only for the first few values of $j \leqslant 2$. For $j > 2$ the solutions become numerical, and the numerical errors grow rapidly. The solutions take simpler forms and the precision becomes better for $A \gg q$, in which case B takes the value

$$B \approx 2(\lambda + m(L-A+m+1)) - m(2L+8j+5)qA. \tag{15}$$

In this approximation the QES Pöschl-Teller potential in (7) becomes comparable with the perturbed Pöschl-Teller potential, a specific form of which has been studied in [15].

GENERALIZED QES PÖSCHL-TELLER POTENTIAL

In this section we present a procedure that relates the QES Pöschl-Teller potential to the *generalized* QES Pöschl-Teller potential. It is amusing to observe that when the coordinate x of the Pöschl-Teller potential is replaced by $x \to x/2$, the QES Pöschl-Teller potential transforms to the generalized QES Pöschl-Teller potential:

$$\begin{aligned} V(x) \;=\; & \frac{\alpha^2}{2}[(L(L+1)+A(A+1))\csc h^2\alpha x \\ & +(L-A)(L+A+1)\coth\alpha x \csc h\alpha x] \\ & +q(2BA^2\sinh^2\frac{\alpha x}{2}+qA^4\sinh^4\frac{\alpha x}{2})\alpha^2\tanh^2\frac{\alpha x}{2} \end{aligned} \tag{16}$$

Obviously when $q = 0$ this potential reduces to the exactly solvable generalized Pöschl-Teller potential. The wave function corresponding to the solution of (5) with the potential of (16) now takes the form

$$\psi(x) = (\cosh\frac{\alpha x}{2})^{qA^2-B-L-4j-2}(\sinh\frac{\alpha x}{2})^{1+L}e^{-\frac{1}{4}qA^2\cosh\alpha x}\Re_j(-\sinh^2\frac{\alpha x}{2}). \tag{17}$$

The corresponding energies are determined as

$$E' = \frac{E}{4} + m(B+4j-m+1)\alpha^2 \tag{18}$$

where E is the eigenvalue of the Schrödinger equation with the QES Pöschl-Teller potential given by(7). It is clear that the same recurrence relation in(11) for the polynomial $P_m(\lambda)$ holds true.

QES PT SYMMETRIC SCARF POTENTIAL

It is interesting to observe that one can transform the QES Pöschle-Teller potential to the PT symmetric QES Scarf potential by replacing x by the complex variable

$$x \to \frac{x}{2} + \frac{i\pi}{4\alpha} \tag{19}$$

Then one obtains the following potential:

$$\begin{aligned} V(x) = & -\frac{\alpha^2}{2}[(L(L+1)+A(A+1))\sec h^2\alpha x \\ & + i(L-A)(L+A+1)\tanh\alpha x \sec h\alpha x] \\ & + \frac{\alpha^2}{4}(2qBA^2\sinh^2(\frac{\alpha x}{2}+\frac{i\pi}{4}) + q^2A^4\sinh^4(\frac{\alpha x}{2}+\frac{i\pi}{4}))\tanh^2(\frac{\alpha x}{2}+\frac{i\pi}{4}) \end{aligned} \tag{20}$$

This is the QES form of the PT symmetric Scarf potential which has not been discussed in the literature. The exactly solvable part of the potential (20)(given in the [...] which, can be obtained if $q = 0$) has been discussed recently by Bagchi[14]. The energy eigenvalues are the same as in (18) obtained for the generalized QES Pöschle-Teller potential. But the wave function now reads

$$\begin{aligned}\psi(x) \;&=\; \left(\frac{i+\tanh\alpha x}{i-\tanh\alpha x}\right)^{\frac{1}{2}(B+2L-qA^2+4j+3)} (\cosh\alpha x)^{\frac{1}{2}(qA^2-4j-B-1)} \times \\ &\quad e^{-\frac{i}{4}qA^2\sinh\alpha x}\Re_j(-\sinh(\frac{\alpha x}{2}+\frac{i\pi}{4}). \end{aligned} \tag{21}$$

Here again the polynomial $P_m(\lambda)$ satisfies the same recurrence relation defined in(11)

The method described in this section in order to transform one type of potential to another type may be further generalized. The transformation $x \to ax+ib$, where a and b, are arbitrary real parameters, preserves the PT symmetry. The energy eigenvalues of the new potential involves further terms in addition to the scaled energy eigenvalues of the QES Pöschl-Teller potential. The corresponding wave function is obtained from the former wave function of the Pöschl-Teller potential by replacing $x \to ax+ib$, and the polynomial $P_m(\lambda)$ obeys the same recurrence relation of(11). This transformation has been discussed in the literature[15] for the exactly solvable potentials.

THE SEXTIC OSCILLATOR

In this section we discuss a method about the transformation of the QES Pöschle-Teller potential to the sextic oscillator potential. The QES Pöschle-Teller potential can be converted to the radial sextic oscillator potential by redefining the parameters and taking suitable limits while keeping the variable x intact. If we redefine the parameters in (6) and (7) by introducing

$$B = \frac{b}{\alpha^2}, \quad A = \frac{a}{\alpha^2}, \quad \lambda = \frac{\varepsilon - mb}{\alpha^2} \tag{22}$$

and taking the limit of the $(V(x) - E)$ term in the Schrödinger equation for $\alpha \to 0$ the Pöschle-Teller potential transforms to the radial sextic harmonic oscillator potential:

$$V = \frac{L(L+1)}{x^2} + (b^2 - (2L+8j+5)qa^2)x^2 + 2bqa^2x^4 + q^2a^4x^6. \tag{23}$$

The ground state wave function(4) takes the form

$$\psi(x) = x^{1+L}e^{-\frac{b}{2}x^2-\frac{q}{4}x^4} \tag{24}$$

and the energy eigenvalue is given by

$$E = 4\varepsilon + (2L+3)b. \tag{25}$$

The general solutions of the Schrödinger equation with the sextic oscillator potential can be obtained by substituting

$$\psi(x) = x^{1+L}e^{-\frac{b}{2}x^2-\frac{q}{4}x^4}\Re_j(x^2) \tag{26}$$

into the Schrödinger equation(5). By rearranging the terms one can obtain,

$$\Re_j(x^2) = \sum_{m=0}^{2j} \frac{(2j)!(2L+1)!(L+m)!}{2m!(2j-m)!(2L+1+2m)!} P_m(\varepsilon)(4qa^2x^2)^m \tag{27}$$

along with the recurrence relation satisfied by $P_m(\varepsilon)$

$$2(2j-m)qa^2P_{m+1}(\varepsilon) + 2(\varepsilon - bm)P_m(\varepsilon) - m(2L+2m+1)P_{m-1}(\varepsilon) = 0. \tag{28}$$

The first four polynomials are given by

$$\begin{aligned}
P_1(\varepsilon) &= \varepsilon \\
P_2(\varepsilon) &= \varepsilon^2 - b\varepsilon - \frac{1}{2}(3+2L)qa^2 \\
P_3(\varepsilon) &= \varepsilon^3 - 3b\varepsilon^2 - 2(b^2 - 2(L+2))\varepsilon + 2b(3+2L)qa^2 \\
P_4(\varepsilon) &= \varepsilon^4 - 6b\varepsilon^3 + (11b^2 - 5(2L+5)qa^2\varepsilon^2 + 3b(-2b^2 + (10L+21)qa^2\varepsilon \\
&\quad + 9\left[-b^2(2L+3) + \frac{1}{4}(4L(5L+1)+21)q^2a^4\right].
\end{aligned} \tag{29}$$

They agree with the polynomials given in references [7] and [10]. We have also compared our results with the numerical solutions obtained in the reference[16], for the potentials,

$$\begin{aligned}
V_1(x) &= x^2 + \frac{x^4}{2(7.625)^{3/2}} + \frac{x^6}{7442} \\
V_2(x) &= x^2 + \frac{x^4}{2(7.375)^{3/2}} + \frac{x^6}{6962} \\
V_3(x) &= x^2 + \frac{x^4}{2(7.125)^{3/2}} + \frac{x^6}{6498}
\end{aligned} \tag{30}$$

where we have obtained exactly the same results of[16], for the energy eigenvalues

$$E_1 = 2.897143, \quad E_2 = 5.891677, \quad and \quad E_3 = 8.991223. \tag{31}$$

corresponding to the potentials $V_1(x), V_2(x), and\ V_3(x)$respectively.

PT SYMMETRIC ANHARMONIC OSCILLATOR POTENTIAL

In order to transform the potential given in (20) to the PT symmetric anharmonic oscillator potential we replace q and L by

$$q \to \frac{q}{A^2}, \quad L \to \frac{1}{8}\left(qL^2 + (2+4A)L - (20-32j)\right). \tag{32}$$

Further, substituting in the parameters in (18) and (20)

$$\begin{aligned}
A &= \sqrt{17}\frac{qa^2}{\alpha^3} + \frac{7}{\sqrt{17}}\frac{b}{\alpha^2}, \quad L = \frac{1}{2}\left(3 - \sqrt{17} + \left(1 - \frac{7}{\sqrt{17}}\right)\frac{\alpha b}{qa^2}\right) \\
\lambda &= \frac{\varepsilon + 2jb}{\alpha^2} + \frac{4jqa^2}{\alpha^3}, q \to \frac{4qa^2}{\alpha^3} - \frac{1}{17}\left(1 + \frac{7}{\sqrt{17}}\right)\frac{2b^2 + 17\ell qa^2}{qa^2\alpha}
\end{aligned} \tag{33}$$

and taking the limit of the $(V(x) - E)$ given in (18) and(20) when $\alpha \to 0$, we obtain the PT symmetric anharmonic oscillator potential:

$$V = 2i(b\ell - (1+2j)qa^2)x + (b^2 - 2\ell qa^2)x^2 + 2iqba^2x^3 - q^2a^4x^4. \tag{34}$$

This leads to the energy eigenvalues,

$$E = \varepsilon + b(1+2j) + \ell^2. \tag{35}$$

The ground state wave function of the potential can be obtained from (21) by using the same limiting procedure as introduced in (32) and (33):

$$\psi = e^{(-ix - \frac{1}{2}bx^2 - \frac{2i}{3}qA^2x^3)} \tag{36}$$

The wave function for any j can be obtained by letting

$$\psi(x) = e^{(-ix - \frac{1}{2}bx^2 - \frac{2i}{3}qA^2x^3)} \sum_{m=0}^{2j} a_m x^{2m}. \tag{37}$$

Here we obtain a four-term recurrence relation for the energy,

$$2i(2j-m)qa^2P_{m+1}(\varepsilon)+(\varepsilon-2b(m-j))P_m(\varepsilon)-2im\ell P_{m-1}(\varepsilon)+m(m-1)P_{m-2}(\varepsilon)=0 \tag{38}$$

The first four P_m is given by

$$\begin{aligned}
P_1(\varepsilon) &= \varepsilon \\
P_2(\varepsilon) &= \varepsilon^2-b^2-4qa^2\ell \\
P_3(\varepsilon) &= \varepsilon^3-4(b^2+4qa^2\ell)\varepsilon-16q^2a^4 \\
P_4(\varepsilon) &= \varepsilon^4-10(b^2+4qa^2l)\varepsilon^2-96q^2a^4\varepsilon+9(b^4+8qb^2a^2\ell+16q^2a^4\ell^2)
\end{aligned} \tag{39}$$

The results are in agreement with those given in[17].

CONCLUSIONS

We have developed a general procedure to obtain the eigenstates of some QES potentials in terms of the orthogonal polynomials as well as the eigenvalues. We have proven that these potentials can be obtained from each other by adequate transformations. Some examples have been presented to test the validity of the procedure given here.

The present work can be generalized in various directions. The method we have introduced can be applied to determine the QES forms of the other exactly solvable potentials. The present construction seems to exhibit closer similarity to the current perturbation constructions. In particular, the perturbed Pöschl-Teller potential can be worked out by following the procedure discussed here and some of its eigenstates can be determined exactly.

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